THE THEORY OF THE NONLINEAR SCHRÖDINGER EQUATION BY PIERRE RAPHAËL AND GIGLIOLA STAFFILANI

1. Course Description

In the last 15 years there as been an incredible number of advances in the general study of dispersive equations. In this course we will have time to only present some of these advances in the context of the nonlinear Schrödinger (NSL) equation in $\mathbf{R}^n$. Nevertheless we hope that the mathematical methods introduced will represent a valuable set of tools for any young mathematician to acquire.

The first part of the course will concern with existence, uniqueness and continuous dependence with respect to the initial data (well-posedness) for the NLS. We start with some by now classical estimates for the solutions of the linear Schrödinger equation and in particular we introduce the Strichartz estimates. We then proceed to the study of the local well-posedness for the nonlinear equation. The energy subcritical case will be attacked first. Here the extension to global solutions for the defocusing or small data problem is relatively simple. In this context we also introduce the "I-method". We then pass to the energy critical problem. This involves a variety of more sophisticated tools and it will bring us to the end of the first part of the course. If some time remains we will introduce few results relative to the periodic NLS.

The second part of the course will be devoted to a qualitative description of the solution in the focusing case. We will briefly recall the standard variational theory of ground states which provides the orbital stability of exceptional periodic solutions: the solitary waves. We will then discuss the possibility of singularity formation and focus onto the so called L^2 critical case. Here we will discuss some recent qualitative results on the singularity formation and make connections which other nonlinear dispersive equations like the critical KdV or the Zakharov system. We will in particular stress the role played by rigidity theorems and classification results of specific nonlinear objects. These objects underly recent breakthrough works both for the study of the finite time blow up dynamics and the derivation of global existence results.

2. Syllabus (15 Lectures)

(1) The Linear Schrödinger Equation in $\mathbf{R}^n$: dispersive and Strichartz estimates (ref: [2]).
(2) The Nonlinear Schrödinger Equation (NLS) in $\mathbf{R}^n$: conservation laws, classical Morawetz estimates, invariances for the equation (ref: [13]).
(3) Local well-posedness for the $H^1(\mathbf{R}^n)$ subcritical NLS (ref: [2] and [13]).
(4) Global well-posedness for the $H^1(\mathbf{R}^n)$ subcritical NLS and the "I-method" (ref: [2], [3] and [13]).
(5) Interaction Morawetz estimates and scattering (ref: [3] and [13]).
(6) Global well-posedness for the $H^1(\mathbf{R}^n)$ critical NLS: Part I.
(7) Global well-posedness for the $H^1(\mathbf{R}^n)$ critical NLS: Part II (ref: [13]).
(8) The periodic NLS (ref: [1]).
(9) Orbital stability of ground states and finite time blow up for the focusing (NLS) (ref: [5], [2])
(10) The concentration phenomenon in the L^2 critical case. Classification of the critical mass blow up solution (ref: [7] and [4]).
(11) Blow up for the critical KdV: a rigidity theorem (ref: [8]).
(12) Blow up close to the ground state: the viriel dispersion (ref: [9]).
(13) Non existence of self similar solutions near the ground state (ref: [10]).
(14) The stable log-log regime (ref: [11])
(15) Blow up of the critical norm in the super critical case: another rigidity theorem (ref: [12]).

References

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