Stability and blow up for the non linear Schrödinger Equation

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Lecture 1

We continue in these notes the study of the power nonlinearity Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbf{R}^N)\}$ in dimension $N \geq 1$ and for energy subcritical nonlinearities:

$$1 < p < +\infty \quad \text{for} \quad N = 1, 2, \quad 1 < p < 2^* - 1 \quad \text{for} \quad N \geq 3. \tag{2}$$

Here $2^* = \frac{2N}{N-2}$ is the Sobolev exponent of the injection $\dot{H}^1 \hookrightarrow L^{2^*}$. Let us recall that the case $p = 3$ appears in various areas of physics: for the propagation of waves propagating in non linear media and optical fibers for $N = 1$, the focusing of laser beams for $N = 2$, the Bose-Einstein condensation phenomenon for $N = 3$, see the monograph [13] for a more systematic introduction to this physical aspect of the problem.

Our aim is to focus onto the description of the long time behavior or the singularity formation of solutions in the energy space H^1. The possibility of finite time blow up corresponding to a self focusing of the nonlinear wave will be of particular interest to us. Note that (NLS) is an infinite dimensional Hamiltonian system without any space localization property and infinite speed of propagation. It is in this context together with the critical generalized KdV equation one of the few examples where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.*

The theory of global existence or blow up for (NLS) as known up to now is intimately connected to the theory of ground states or solitons which are special periodic solutions to the Hamiltonian system. A central question is the stability of these solutions and the description of the flow around them which has attracted a considerable amount of work for the past thirty years. Here we shall introduce a new angle of attack for these problems based on *Liouville* type theorems and the *dynamical* classification of the soliton solution among the solutions to the Hamiltonian system.

These notes are organized as follows.

In the first section, we recall the main standard results about subcritical non linear Schrödinger equations and in particular the existence and orbital stability of soliton like solutions. In the second section, we focus onto the critical blow up problem and recall the few general results known on the singularity formation in this case. Section 3 is devoted to an exposition of the series of results obtained in collaboration with F.Merle in [7], [8], [9], [10], [11] and [12] and which allow in particular a complete description of the so-called stable log-log blow up dynamics. In section 4, we present a detailed proof of the first of these results which provides a sharp upper bound on blow up rate for a suitable class of initial data. In section 5, we outline the main steps of the proof of the sharp lower bound on the blow up rate and the mass quantization theorem which rely on fine dispersive properties of the flow. We expect the presentation to be essentially self contained provided the prior knowledge of standard tools in the study of non linear PDE's.

1 The subcritical problem

In this section, we recall the main classical facts regarding the global well posedness in the energy space of (NLS). We will in particular introduce a fundamental object for the study of (1): the ground state solitary wave.

1.1 Global well posedness in the subcritical case

Let us consider the general non linear Schrödinger equation:

$$\begin{cases} iu_t = -\Delta u - |u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1 \end{cases} \tag{3}$$

with p satisfying the energy subcriticality assumption (2). The local well posedness of (3) in H^1 is a result of Ginibre, Velo, [3], see also [4]. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$. Moreover, the life time of the solution can be proved to be lower bounded by a function depending on the H^1 size of the solution only, $T(u_0) \geq f(\|u_0\|_{H^1})$, and hence there holds the blow up alternative:

$$T < +\infty \ \text{ implies } \ \lim_{t\to T} \|u(t)\|_{H^1} = +\infty. \tag{4}$$

To prove the global existence of the solution, it thus suffices to control the size of the solution in H^1. This is achieved in some cases using the invariants of the flow. Indeed, the following H^1 quantities are conserved by the flow:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{5}$$

- Energy -or Hamiltonian-:
$$E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0); \tag{6}$$

- Momentum:
$$Im\left(\int \nabla u\overline{u}(t,x)\right) = Im\left(\int \nabla u_0\overline{u_0}(x)\right). \tag{7}$$

Note that the growth condition on the non linearity (2) ensures from Sobolev embedding that the energy is well defined, and this is why H^1 is referred to as the energy space.

These invariants are related to the group of symmetry of (3) in H^1:

- Space-time translation invariance: if $u(t,x)$ solves (3), then so does $u(t+t_0, x+x_0)$, $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^N$.
- Phase invariance: if $u(t,x)$ solves (3), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbf{R}$.
- Scaling invariance: if $u(t,x)$ solves (3), then so does $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: if $u(t,x)$ solves (3), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}$, $\beta \in \mathbf{R}^N$.

Let us point out that this group of H^1 symmetries is the same like for the *linear* Schrödinger equation -up to the conformal invariance to which we will come back later-.

The *critical space* is defined to be the Sobolev space which is invariant by the scaling symmetry:
$$|u_\lambda(t)|_{\dot{H}^{s_c}} = |u(\lambda^2 t)|_{\dot{H}^{s_c}} \quad \text{for} \quad s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{8}$$
Note that $s_c < 1$ from (2).

As an outcome, we have the following result:

Theorem 1 (Global wellposedness in the subcritical case) *Let $N \geq 1$ and $1 < p < 1 + \frac{4}{N}$ -equivalently $s_c < 0$-, then all solutions to (3) are global and bounded in H^1.*

Proof of Theorem 1

The proof relies on an a priori bound on the H^1 norm of the solution. Indeed, $|u(t)|_{L^2} = |u_0|_{L^2}$. Next, there holds the Gagliardo-Nirenberg interpolation estimate:

$$\forall v \in H^1, \quad \int |v|^{p+1} \leq C(N,p) \left(\int |\nabla v|^2 \right)^{\frac{N(p-1)}{4}} \left(\int |v|^2 \right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}}. \tag{9}$$

Applying this with $v = u(t)$, we get from the conservation of the energy and the L^2 norm:

$$\forall t \in [0,T), \quad E_0 \geq \frac{1}{2}[\int |\nabla v|^2 - C(u_0) \left(\int |\nabla v|^2 \right)^{\frac{N(p-1)}{4}}].$$

The subcriticality assumption $p < 1 + \frac{4}{N}$ now implies an a priori bound on the H^1 norm which concludes the proof of Theorem 1.

The critical exponent

$$p = 1 + \frac{4}{N} \quad \text{ie} \quad s_c = 0$$

arises from this analysis and corresponds to the so-called L^2 *critical case.* It is the smallest power nonlinearity for which blow up can occur and corresponds to an exact balance between the kinetic and potential energies under the constraint of conserved L^2 mass. The L^2 supercritical -and energy subcritical cases- corresponds to

$$1 + \frac{4}{N} < p < 2^* - 1 \quad \text{ie} \quad 0 < s_c < 1.$$

1.2 The solitary wave

We focus in this subsection onto the subcritical case

$$1 < p < 1 + \frac{4}{N} \tag{10}$$

and aim at understanding the long time dynamics of the flow.
A fundamental feature of the focusing (NLS) problem is the existence of periodic solutions. Indeed,

$$u(t,x) = \phi(x)e^{it}$$

is an H^1 solution to (3) iff ϕ solves the nonlinear elliptic equation:

$$\Delta\phi - \phi + \phi|\phi|^{p-1} = 0, \quad \phi \in H^1(\mathbf{R}^N) \tag{11}$$

There are plenty of ways to construct solutions to (11), the simplest of which being to look for radial solutions and using a shooting method, see [1].

Proposition 1 (Existence of radial profiles to (11)) *Let $N \geq 1$ and p satisfy (2), then there exists radially symmetric solutions to (11). In fact, the following holds:*
(i) For $N = 1$, all solutions to (11) are translates of

$$Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{(p-1)x}{2})} \right)^{p-1}. \tag{12}$$

(ii) For $N \geq 2$, there exist a sequence of radial solutions $(Q_n)_{n\geq 0}$ with increasing L^2 norm such that Q_n vanishes n times on $\mathbf{R}^N$.

The exact structure of the set of solutions to (11) is not known in dimension $N \geq 2$. An important rigidity property however which combines nonlinear elliptic techniques and ODE techniques is *the uniqueness of the nonnegative* solution $Q = Q_0$ to (11).

Proposition 2 (Uniqueness of the ground state) *All solutions to*

$$\Delta\phi - \phi + \phi|\phi|^{p-1} = 0, \quad \phi \in H^1(\mathbf{R}^N), \quad \phi(x) > 0 \tag{13}$$

are a translate of an exponentially decreasing $\mathcal{C}^2$ radial profile $Q(r)$ ([2]) which is the unique nonnegative radially symmetric solution to (11) ([5]). Q is the socalled ground state solution.

The uniqueness is thus the consequence of two facts: a positive decaying at infinity solution to (13) is necessarily radially symmetric with respect to a point. This is a very deep and non trivial result due to Gidas, Ni, Nirenberg [2] and which relies on the maximum principle. Then there is uniqueness of the radial decaying positive solution in the ODE sense. The original -and delicate- proof of this last fact by Kwong [5] has been revisited by MacLeod [6] and is very nicely presented in the Appendix of Tao [14].

Let us now observe that we may let the full group of symmetries of (3) act on the solitary wave $u(t,x) = Q(x)e^{it}$ to get a $2N+2$ parameters family of solitary waves:

$$u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda(x+x_0) - \lambda^2\beta t)e^{i\lambda^2 t}e^{i\gamma_0}e^{i\frac{\beta}{2}\cdot(\lambda(x+x_0)-\lambda^2\beta t)}, \;\; (\lambda, x_0, \gamma_0, \beta) \in \mathbf{R}_+^* \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N.$$

Observe that these waves are moving according to the free Galilean motion and oscillating at a phase related to their size λ: the larger the λ, the wilder the oscillations in time. Observe also that the fact the problem is subcritical in both L^2 and H^1 implies that a solitary wave can be made *arbitrarily small* in H^1.

Let us finish this section by introducing the Korteweg de Vries equation for which the structure of the solitary wave family is somehow more enlightening. The KdV system corresponds to the description of waves propagating at the surface of water in certain regimes. It is the case $p = 2$ of the generalized KdV equations:

$$(gKdV) \quad \begin{cases} u_t + (u_{xx} + u^p)_x = 0, \quad (t,x) \in [0,T) \times \mathbf{R}, \\ u(0,x) = u_0(x), \quad u_0 : \mathbf{R} \to \mathbf{R}. \end{cases} \tag{14}$$

This system is also a nonlinear dispersive equation and admits the same two conservation laws like the (NLS):

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

$$E(u) = \frac{1}{2}\int |u_x|^2 - \frac{1}{p+1}\int |u|^{p+1} = E(u_0).$$

The Cauchy problem is moreover subcritical in H^1 and thus all solutions are global and bounded in H^1 for $p < 5$. We now look for traveling waves propagating at speed $c > 0$: $u(t,x) = Q_c(x - ct)$ solves (14) iff

$$(Q_c)_{xx} - cQ_c + Q_c^p = 0, \quad Q_c \in H^1.$$

By rescaling, this implies :

$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$$

where Q is the H^1 solution to (11) which is in fact explicit and given by (12). An explicit computation then yields:

$$\|Q_c\|_{L^2} = c^{\frac{5-p}{8(p-1)}} \|Q\|_{L^2}, \quad \|\nabla Q_c\|_{L^2} = c^{\frac{3p+1}{8(p-1)}} \|\nabla Q\|_{L^2}.$$

The outcome is that small solitons travel slowly while large solitons are fast.

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