

## Linear Algebraic Groups

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ABSTRACT. We give a summary, without proofs, of basic properties of linear algebraic groups, with particular emphasis on reductive algebraic groups.

### 1. Algebraic groups

Let  $K$  be an algebraically closed field. An *algebraic  $K$ -group*  $\mathbf{G}$  is an algebraic variety over  $K$ , and a group, such that the maps  $\mu : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$ ,  $\mu(x, y) = xy$ , and  $\iota : \mathbf{G} \rightarrow \mathbf{G}$ ,  $\iota(x) = x^{-1}$ , are morphisms of algebraic varieties. For convenience, in these notes, we will fix  $K$  and refer to an algebraic  $K$ -group as an algebraic group. If the variety  $\mathbf{G}$  is affine, that is,  $\mathbf{G}$  is an algebraic set (a Zariski-closed set) in  $K^n$  for some natural number  $n$ , we say that  $\mathbf{G}$  is a *linear algebraic group*. If  $\mathbf{G}$  and  $\mathbf{G}'$  are algebraic groups, a map  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  is a *homomorphism of algebraic groups* if  $\varphi$  is a morphism of varieties and a group homomorphism. Similarly,  $\varphi$  is an *isomorphism of algebraic groups* if  $\varphi$  is an isomorphism of varieties and a group isomorphism.

A closed subgroup of an algebraic group is an algebraic group. If  $\mathbf{H}$  is a closed subgroup of a linear algebraic group  $\mathbf{G}$ , then  $\mathbf{G}/\mathbf{H}$  can be made into a quasi-projective variety (a variety which is a locally closed subset of some projective space). If  $\mathbf{H}$  is normal in  $\mathbf{G}$ , then  $\mathbf{G}/\mathbf{H}$  (with the usual group structure) is a linear algebraic group.

Let  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  be a homomorphism of algebraic groups. Then the kernel of  $\varphi$  is a closed subgroup of  $\mathbf{G}$  and the image of  $\varphi$  is a closed subgroup of  $\mathbf{G}'$ .

Let  $X$  be an affine algebraic variety over  $K$ , with affine algebra (coordinate ring)  $K[X] = K[x_1, \dots, x_n]/I$ . If  $k$  is a subfield of  $K$ , we say that  $X$  is *defined over  $k$*  if the ideal  $I$  is generated by polynomials in  $k[x_1, \dots, x_n]$ , that is,  $I$  is generated by  $I_k := I \cap k[x_1, \dots, x_n]$ . In this case, the  $k$ -subalgebra  $k[X] := k[x_1, \dots, x_n]/I_k$  of  $K[X]$  is called a  *$k$ -structure* on  $X$ , and  $K[X] = k[X] \otimes_k K$ . If  $X$  and  $X'$  are algebraic varieties defined over  $k$ , a morphism  $\varphi : X \rightarrow X'$  is *defined over  $k$*  (or is a  *$k$ -morphism*) if there is a homomorphism  $\varphi_k^* : k[X'] \rightarrow k[X]$  such that the algebra homomorphism  $\varphi^* : K[X'] \rightarrow K[X]$  defining  $\varphi$  is  $\varphi_k^* \times id$ . Equivalently, the coordinate functions of  $\varphi$  all have coefficients in  $k$ . The set  $X(k) := X \cap k^n$  is called the  *$K$ -rational points* of  $X$ .

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If  $k$  is a subfield of  $K$ , we say that a linear algebraic group  $\mathbf{G}$  is *defined over  $k$*  (or is a  *$k$ -group*) if the variety  $\mathbf{G}$  is defined over  $k$  and the homomorphisms  $\mu$  and  $\iota$  are defined over  $k$ . Let  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  be a  $k$ -homomorphism of  $k$ -groups. Then the image of  $\varphi$  is defined over  $k$  but the kernel of  $\varphi$  might not be defined over  $k$ .

An algebraic variety  $X$  over  $K$  is *irreducible* if it cannot be expressed as the union of two proper closed subsets. Any algebraic variety  $X$  over  $K$  can be expressed as the union of finitely many irreducible closed subsets:

$$X = X_1 \cup X_2 \cup \cdots \cup X_r,$$

where  $X_i \not\subset X_j$  if  $j \neq i$ . This decomposition is unique and the  $X_i$  are the maximal irreducible subsets of  $X$  (relative to inclusion). The  $X_i$  are called the *irreducible components* of  $X$ .

Let  $\mathbf{G}$  be an algebraic group. Then  $\mathbf{G}$  has a unique irreducible component  $\mathbf{G}^0$  containing the identity element. The irreducible component  $\mathbf{G}^0$  is a closed normal subgroup of  $\mathbf{G}$ . The cosets of  $\mathbf{G}^0$  in  $\mathbf{G}$  are the irreducible components of  $\mathbf{G}$ , and  $\mathbf{G}^0$  is the connected component of the identity in  $\mathbf{G}$ . Also, if  $\mathbf{H}$  is a closed subgroup of  $\mathbf{G}$  of finite index in  $\mathbf{G}$ , then  $\mathbf{H} \supset \mathbf{G}^0$ . For a linear algebraic group, connectedness is equivalent to irreducibility. It is usual to refer to an irreducible algebraic group as a connected algebraic group.

If  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  is a homomorphism of algebraic groups, then  $\varphi(\mathbf{G}^0) = \varphi(\mathbf{G})^0$ . If  $k$  is a subfield of  $K$  and  $\mathbf{G}$  is defined over  $k$ , then  $\mathbf{G}^0$  is defined over  $k$ .

The *dimension* of  $\mathbf{G}$  is the dimension of the variety  $\mathbf{G}^0$ . That is, the dimension of  $\mathbf{G}$  is the transcendence degree of the field  $K(\mathbf{G}^0)$  over  $K$ .

If  $\mathbf{G}$  is a linear algebraic group, then  $\mathbf{G}$  is isomorphic, as an algebraic group, to a closed subgroup of  $\mathbf{GL}_n(K)$  for some natural number  $n$ .

**EXAMPLE 1.1.**  $\mathbf{G} = K$ , with  $\mu(x, y) = x + y$  and  $\iota(x) = -x$ . The usual notation for this group is  $\mathbf{G}_a$ . It is connected and has dimension 1.

**EXAMPLE 1.2.** Let  $n$  be a positive integer and let  $M_n(K)$  be the set of  $n \times n$  matrices with entries in  $K$ . The general linear group  $\mathbf{G} = \mathbf{GL}_n(K)$  is the group of matrices in  $M_n(K)$  that have nonzero determinant. Note that  $\mathbf{G}$  can be identified with the closed subset  $\{(g, x) \mid g \in M_n(K), x \in K, (\det g)x = 1\}$  of  $K^{n^2} \times K = K^{n^2+1}$ . Then  $K[\mathbf{G}] = K[x_{ij}, 1 \leq i, j \leq n, \det(x_{ij})^{-1}]$ . The dimension of  $\mathbf{GL}_n(K)$  is  $n^2$ , and it is connected. In the case  $n = 1$ , the usual notation for  $\mathbf{GL}_1(K)$  is  $\mathbf{G}_m$ . The only connected algebraic groups of dimension 1 are  $\mathbf{G}_a$  and  $\mathbf{G}_m$ .

**EXAMPLE 1.3.** Let  $n$  be a positive integer and let  $I_n$  be the  $n \times n$  identity matrix. The  $2n \times 2n$  matrix  $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$  is invertible and satisfies  ${}^t J = -J$ , where  ${}^t J$  denotes the transpose of  $J$ . The  $2n \times 2n$  symplectic group  $\mathbf{G} = \mathbf{Sp}_{2n}(K)$  is defined by  $\{g \in M_{2n}(K) \mid {}^t g J g = J\}$ .

## 2. Jordan decomposition in linear algebraic groups

Recall that a matrix  $x \in M_n(K)$  is *semisimple* if  $x$  is diagonalizable: there is a  $g \in \mathbf{GL}_n(K)$  such that  $g x g^{-1}$  is a diagonal matrix. Also,  $x$  is unipotent if  $x - I_n$  is nilpotent:  $(x - I_n)^k = 0$  for some natural number  $k$ . Given  $x \in \mathbf{GL}_n(K)$ , there exist elements  $x_s$  and  $x_u$  in  $\mathbf{GL}_n(K)$  such that  $x_s$  is semisimple,  $x_u$  is unipotent, and  $x = x_s x_u = x_u x_s$ . Furthermore,  $x_s$  and  $x_u$  are uniquely determined.

Now suppose that  $\mathbf{G}$  is a linear algebraic group. Choose  $n$  and an injective homomorphism  $\varphi : \mathbf{G} \rightarrow \mathbf{GL}_n(K)$  of algebraic groups. If  $g \in \mathbf{G}$ , the semisimple

and unipotent parts  $\varphi(g)_s$  and  $\varphi(g)_u$  of  $\varphi(g)$  lie in  $\varphi(\mathbf{G})$ , and the elements  $g_s$  and  $g_u$  such that  $\varphi(g_s) = \varphi(g)_s$  and  $\varphi(g_u) = \varphi(g)_u$  depend only on  $g$  and not on the choice of  $\varphi$  (or  $n$ ). The elements  $g_s$  and  $g_u$  are called the semisimple and unipotent part of  $g$ , respectively. An element  $g \in \mathbf{G}$  is *semisimple* if  $g = g_s$  (and  $g_u = 1$ ), and *unipotent* if  $g = g_u$  (and  $g_s = 1$ ).

*Jordan decomposition.*

- (1) If  $g \in \mathbf{G}$ , there exist elements  $g_s$  and  $g_u$  in  $\mathbf{G}$  such that  $g = g_s g_u = g_u g_s$ ,  $g_s$  is semisimple, and  $g_u$  is unipotent. Furthermore,  $g_s$  and  $g_u$  are uniquely determined by the above conditions.
- (2) If  $k$  is a perfect subfield of  $K$  and  $\mathbf{G}$  is a  $k$ -group, then  $g \in \mathbf{G}(k)$  implies  $g_s, g_u \in \mathbf{G}(k)$ .

Jordan decompositions are preserved by homomorphisms of algebraic groups. Suppose that  $\mathbf{G}$  and  $\mathbf{G}'$  are linear algebraic groups and  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  is a homomorphism of linear algebraic groups. Let  $g \in \mathbf{G}$ . Then  $\varphi(g)_s = \varphi(g_s)$  and  $\varphi(g)_u = \varphi(g_u)$ .

### 3. Lie algebras

Let  $\mathbf{G}$  be a linear algebraic group. The tangent bundle  $T(\mathbf{G})$  of  $\mathbf{G}$  is the set  $\text{Hom}_{K\text{-alg}}(K[\mathbf{G}], K[t]/(t^2))$  of  $K$ -algebra homomorphisms from the affine algebra  $K[\mathbf{G}]$  of  $\mathbf{G}$  to the algebra  $K[t]/(t^2)$ . If  $g \in \mathbf{G}$ , the evaluation map  $f \mapsto f(g)$  from  $K[\mathbf{G}]$  to  $K$  is a  $K$ -algebra isomorphism. This results in a bijection between  $\mathbf{G}$  and  $\text{Hom}_{K\text{-alg}}(K[\mathbf{G}], K)$ . Composing elements of  $T(\mathbf{G})$  with the map  $a + bt + (t^2) \mapsto a$  from  $K[t]/(t^2)$  to  $K$  results in a map from  $T(\mathbf{G})$  to  $\mathbf{G} = \text{Hom}_{K\text{-alg}}(K[\mathbf{G}], K)$ . The tangent space  $T_1(\mathbf{G})$  of  $\mathbf{G}$  at the identity element 1 of  $\mathbf{G}$  is the fibre of  $T(\mathbf{G})$  over 1. If  $X \in T_1(\mathbf{G})$  and  $f \in K[\mathbf{G}]$ , then  $X(f) = f(1) + t d_X(f) + (t^2)$  for some  $d_X(f) \in K$ . This defines a map  $d_X : K[\mathbf{G}] \rightarrow K$  which satisfies:

$$d_X(f_1 f_2) = d_X(f_1) f_2(1) + f_1(1) d_X(f_2), \quad f_1, f_2 \in K[\mathbf{G}].$$

Let  $\mu^* : K[\mathbf{G}] \rightarrow K[\mathbf{G}] \otimes_K K[\mathbf{G}]$  be the  $K$ -algebra homomorphism which corresponds to the multiplication map  $\mu : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$ . Set  $\delta_X = (1 \otimes d_X) \circ \mu^*$ . The map  $\delta_X : K[\mathbf{G}] \rightarrow K[\mathbf{G}]$  is a  $K$ -linear map and a *derivation*:

$$\delta_X(f_1 f_2) = \delta_X(f_1) f_2 + f_1 \delta_X(f_2), \quad f_1, f_2 \in K[\mathbf{G}].$$

Furthermore,  $\delta_X$  is left-invariant:  $\ell_g \delta_X = \delta_X \ell_g$  for all  $g \in \mathbf{G}$ , where  $(\ell_g f)(g') = f(g^{-1} g')$ ,  $f \in K[\mathbf{G}]$ . The map  $X \mapsto \delta_X$  is a  $K$ -linear isomorphism of  $T_1(\mathbf{G})$  onto the vector space of  $K$ -linear maps from  $K[\mathbf{G}]$  to  $K[\mathbf{G}]$  which are left-invariant derivations.

Let  $\mathfrak{g} = T_1(\mathbf{G})$ . Define  $[X, Y] \in \mathfrak{g}$  by  $\delta_{[X, Y]} = \delta_X \circ \delta_Y - \delta_Y \circ \delta_X$ . Then  $\mathfrak{g}$  is a vector space over  $K$  and the map  $[\cdot, \cdot]$  satisfies:

- (1)  $[\cdot, \cdot]$  is linear in both variables
- (2)  $[X, X] = 0$  for all  $X \in \mathfrak{g}$
- (3)  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  for all  $X, Y, Z \in \mathfrak{g}$ . (*Jacobi identity*)

Therefore  $\mathfrak{g}$  is a Lie algebra over  $K$ . We call it the Lie algebra of  $\mathbf{G}$ .

EXAMPLE 3.1. If  $\mathbf{G} = \mathbf{GL}_n(K)$ , then  $\mathfrak{g}$  is isomorphic to the Lie algebra  $\mathfrak{gl}_n(K)$  which is  $M_n(K)$  equipped with the Lie bracket  $[X, Y] = XY - YX$ ,  $X, Y \in M_n(K)$ .